Kicking around a binomial sum

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A problem on one of the probability sheets leads (if tackled one way) to the sum,

$$S_r = \sum_{n \geq r} \binom{n}{r} p^n,$$

and our purpose is to evaluate that sum using as many methods as possible, as a way of illustrating some techniques.

1 Generating functions

Let's try setting $G(x) = \sum_{r \geq 0} S_r x^r$. Expanding S_r and interchanging the order of the two summations, we obtain,

$$G(x) = \sum_{n \geq 0} p^n \sum_{0 \leq r \leq n} \binom{n}{r} x^r,$$

and, using the binomial theorem,

$$G(x) = \sum_{n \geq 0} p^n (1+x)^n = \frac{1}{1 - p(1+x)}.$$

We can solve our problem by expanding this last expression as a power series in x and equating coefficients; for that purpose, it's convenient to put the denominator in the form $a(1 - bx)$ and write

$$G(x) = \frac{1}{q(1 - px/q)} = \frac{1}{q} \sum_{r \geq 0} \frac{p^r}{q^r} x^r,$$

where $q = 1 - p$. Thus we obtain the result $S_r = p^r / q^{r+1}$.

2 Negating coefficients

We can extend the conventional definition, $\binom{r}{n} = r!/n!(r-n)!$, to arbitrary values of r by writing

$$\binom{r}{n} = \frac{r(r-1)\ldots(r-n+1)}{n!}, \tag{1}$$

and this makes the binomial theorem hold for negative indices. We then find that

$$\binom{-(r+1)}{n} = \frac{-(r+1)\cdot -(r+2)\cdot \ldots \cdot -(r+n)}{n!} = (-1)^n \binom{r+n}{n}.$$

By shifting the indices a little, and also using the equation, $\binom{n+r}{r} = \binom{n+r}{n}$, we can apply this to our problem:

$$\sum_{n\geq r} \binom{n}{r} p^n = p^r \sum_{n\geq 0} \binom{n+r}{r} p^n = p^r \sum_{n\geq 0} \binom{-(r+1)}{n} (-p)^n.$$

The sum in this last expression is just the binomial expansion of $(1-p)^{-(r+1)}$, and we obtain $S_r = p^r/q^{r+1}$ as before.

3 A recurrence

The relationship $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ allows us to find a recurrence satisfied by S_r. We begin by shifting the index of summation by one step:

$$S_{r+1} = p \sum_n \binom{n+1}{r+1} p^n = p \left(\sum_n \binom{n}{r} p^n + \sum_n \binom{n}{r+1} p^n \right) = p(S_r + S_{r+1}).$$

(Adopting the convention $\binom{n}{r} = 0$ if $r > n$ allows us to drop the limits from the sums and avoid irksome corner cases.) Rearranging gives $qS_{r+1} = pS_r$, and adding the boundary condition $S_0 = 1/q$ gives the solution, $S_r = p^r/q^{r+1}$.

4 Repeated differentiation

Using (1), and including some extra zero terms for $0 \leq n < r$, we find

$$S_r = \frac{p^r}{r!} \sum_{n\geq 0} n(n-1)\ldots(n-r+1)p^{n-r}.$$

But the summand here can be written as the r'th derivative of p^n, and interchanging derivative and summation gives us,

$$S_r = \frac{p^r}{r!} \cdot \frac{d^r}{dp^r} \sum_{n\geq 0} p^n = \frac{p^r}{r!} \cdot \frac{d^r}{dp^r} \frac{1}{1-p}.$$

Differentiating $1/(1-p)$ repeatedly, we obtain

$$\frac{1}{(1-p)^2}, \frac{2}{(1-p)^3}, \frac{6}{(1-p)^4}, \ldots, \frac{r!}{(1-p)^{r+1}}, \ldots$$

Thus

$$S_r = p^r \frac{1}{(1-p)^{r+1}},$$

as before.

5 Another recurrence

Binomial coefficients satisfy the relationship $\binom{n+1}{r+1} = \frac{n+1}{r+1}\binom{n}{r}$, which is easily proved from (1). We can use this to perturb the sum S_r in a different way, starting with a shift of the index of summation:

$$S_{r+1} = \sum_n \binom{n+1}{r+1} p^{n+1} = p \sum_n \frac{n+1}{r+1} \binom{n}{r} p^n.$$

Now we can use $\frac{d}{dp} p^{n+1} = (n+1)p^n$ and interchange derivative and summation to get

$$S_{r+1} = \frac{p}{r+1} \cdot \frac{d}{dp} \sum_n \binom{n}{r} p^{n+1} = \frac{p}{r+1} \cdot \frac{d}{dp} p S_r.$$

As before, we can begin with $S_0 = 1/(1-p)$ and prove by induction that $S_r = p^r/(1-p)^{r+1}$. For the induction step, we calculate the derivative of pS_r as

$$\frac{d}{dp} \left(\frac{p}{1-p} \right)^{r+1} = (r+1) \left(\frac{p}{1-p} \right)^r \cdot \frac{d}{dp} \left(\frac{1}{1-p} - 1 \right) = \frac{(r+1)p^r}{(1-p)^{r+2}}.$$

It follows that $S_{r+1} = p^{r+1}/(1-p)^{r+2}$.