The Twelve Days of Christmas

Mike Spivey, December 2013

Maths teachers love to give themselves a quiet lesson at the end of the Autumn term by asking their classes to count the total number of gifts given by the 'true love' during the twelve days of Christmas. Setting legalistic objections aside (yes, a partridge in a pear tree is one present), and ignoring the IT teacher who writes the program

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sum [ sum [1..n] | n <- [1..12] ]
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(or whatever the equivalent is in Visual Basic), let's try to solve that problem in a properly mathematical way.

Following Knuth, we will use the notation $x^{\underline{n}}$ for the product

$$x(x-1)(x-2)\ldots(x-n+1),$$

and write $\Delta f(x)$ for $f(x+1) - f(x)$. We can immediately calculate that

$$\begin{aligned}\Delta x^{\underline{n}} &= (x+1)x\ldots(x-n+2) - x(x-1)\ldots(x-n+2)(x-n+1)\\ &= [(x+1) - (x-n+1)]x(x-1)\ldots(x-n+2)\\ &= nx^{\underline{n-1}}.\end{aligned}$$

If the Δ notation looks like differentiation, the next one looks like integration. Let's define

$$\textstyle\sum_a^b f(x)\,\delta x = f(a) + f(a+1) + \cdots + f(b-1).$$

We can prove that if $\Delta g(x) = f(x)$ then $\sum_a^b f(x)\,\delta x = g(b) - g(a)$:

$$\begin{aligned}\textstyle\sum_a^b f(x)\,\delta x &= (g(a+1) - g(a)) + (g(a+2) - g(a+1)) + \cdots + (g(b) - g(b-1))\\ &= g(b) - g(a).\end{aligned}$$

So, in particular, $\sum_a^b x^{\underline{n}}\,\delta x = \frac{1}{n+1}(b^{\underline{n+1}} - a^{\underline{n+1}})$. Now, the sum we seek is

$$S_{12} = \textstyle\sum_1^{13}\left(\sum_1^{x+1} y\,\delta y\right)\delta x.$$

But $y = y^{\underline{1}}$, so $\sum_1^{x+1} y\,\delta y = \frac{1}{2}((x+1)^{\underline{2}} - 1^{\underline{2}}) = \frac{1}{2}(x+1)^{\underline{2}}$, and so

$$S_{12} = \textstyle\sum_1^{13} \frac{1}{2}(x+1)^{\underline{2}}\,\delta x = \frac{1}{2}\sum_2^{14} x^{\underline{2}}\,\delta x = \frac{1}{6}(14^{\underline{3}} - 2^{\underline{3}}) = \frac{1}{6}\cdot 2184 = 364.$$

The calculation is easily extended if the festive season should go on for more than twelve days, and generalises naturally if a subsequent 'true love' should offer on each day all the presents that the current one would have offered, up to and including the day in question.

A combinatorial interpretation

The presents for each day can be arranged into a triangle with each row containing presents of one kind; a partridge in a pear tree is at the apex, and twelve drummers drumming, or whatever, at the base. In effect, the triangles assemble to form a tetrahedron where the third dimension is time.

We calculated the answer to the problem as $\frac{1}{3!} \cdot 14^{\underline{3}}$, i.e., as $\binom{14}{3}$, since in general

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}.$$

This is equal to the number of triples (x, u, v) with $1 \le x < u < v \le 14$, or, letting $y = u - 1$ and $z = v - 2$, to the number of triples (x, y, z) with $1 \le x \le y \le z \le 12$. Considering the triangular arrangement, we can use z to choose a day; the triangle for that day will contain z rows, and we can use y to choose one of them, counting from the partridge in a pear tree by way of the two turtle doves to the five go-old rings and beyond. This row will contain y presents, and we can use x to choose a single present. Thus the presents are in one-to-one correspondence with the triples, and the number of presents is 364 as we calculated.

Using generating functions

An alternative approach uses generating functions. If $G(z) = \sum_{n \ge 0} a_n z^n$ is the generating function of a sequence $\langle a_0, a_1, a_2, \ldots \rangle$, then we can obtain the generating function $H(z)$ for the sequence of partial sums

$$\langle a_0, a_0{+}a_1, a_0{+}a_1{+}a_2, \ldots \rangle$$

by multiplying by $1/(1-z)$, so that

$$H(z) = \frac{G(z)}{1-z}.$$

This corresponds to taking the *convolution* of the original sequence $\langle a_n \rangle$ with the sequence $\langle 1, 1, 1, \ldots \rangle$, whose generating function is $1 + z + z^2 + \ldots = 1/(1-z)$. The operation of convolution on sequences acts by multiplying together their generating functions.

We can solve our puzzle by repeatedly taking partial sums:

$$\begin{aligned}
&\langle 1, 1, 1, 1, \ldots \rangle \text{ has generating function } \frac{1}{1-z},\\
&\langle 1, 2, 3, 4, \ldots \rangle \text{ has generating function } \frac{1}{(1-z)^2},\\
&\langle 1, 3, 6, 10, \ldots \rangle \text{ has generating function } \frac{1}{(1-z)^3},\\
&\langle 1, 4, 10, 20, \ldots \rangle \text{ has generating function } \frac{1}{(1-z)^4}.
\end{aligned}$$

Thus, if Christmas has $n+1$ days, we seek the coefficient of z^n in the binomial expansion of $1/(1-z)^4$, namely

$$(-1)^n \binom{-4}{n} = \frac{4 \cdot 5 \cdot \ldots \cdot (n+3)}{n!} = \binom{n+3}{n} = \binom{n+3}{3}.$$

The answer for $n + 1 = 12$ is $\binom{14}{3} = 364$ as before.