Jamie's sum

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Michaelmas Term, 2012

Jamie Frost asked for the sum,

$$S_n = \sum_{1 \le j \le n+1} \sum_{0 \le m \le j-1} \binom{n+1}{j} \binom{n}{m}.$$

The bounds are a bit messy, so we'll begin by rewriting it as

$$S_n = \sum_{0 \le k \le n} \sum_{0 \le m \le k} \binom{n+1}{k+1} \binom{n}{m}.$$

The inner sum is partial, and exchanging the bounds doesn't help with that, so let's be bold and put $m = k - r$:

$$S_n = \sum_{0 \le k \le n} \sum_{0 \le r \le k} \binom{n+1}{k+1} \binom{n}{k-r}.$$

Now interchanging the order of summation is helpful, since

$$S_n = \sum_{0 \le r \le n} \sum_{r \le k \le n} \binom{n+1}{k+1} \binom{n}{k-r},$$

and one or other factor in the summand vanishes when $k < r$ or $k > n$, so we can drop the bounds on k.

Equation (5.23) in *Concrete Math* is a form of Vandermonde's convolution, and holds for $l \ge 0$.

$$\sum_k \binom{l}{k+a} \binom{s}{k+b} = \binom{l+s}{l-a+b}.$$

Applying that to the inner sum, with $l = n+1$, $s = n$, $a = 1$, $b = -r$, we obtain

$$S_n = \sum_{0 \le r \le n} \binom{2n+1}{n-r}.$$

But this is the sum of half a row in Pascal's triangle, so we deduce $S_n = 2^{2n+1}/2 = 2^{2n}$.

A wider perspective

I now learn that the sum S_n arose as the solution to a probability problem: if A flips a coin $n+1$ times and B flips it n times, the probability that A obtains strictly more heads than B is $S_n/2^{n+1} = 1/2$. But there's an easier way to solve the same problem: let A_k be the number of heads obtained by A on the first k flips, and so on, and let $p = P(A_n = B_n)$. Then

$$P(A_{n+1} > B_n) = P(A_n > B_n) + \tfrac{1}{2}P(A_n = B_n) = \tfrac{1}{2}(1-p) + \tfrac{1}{2}p = \tfrac{1}{2}.$$

Let's translate that argument into the language of sums. We have

$$S_n = \sum_{0\le k\le n}\sum_{0\le m\le k}\binom{n+1}{k+1}\binom{n}{m}.$$

Using the recurrence $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ and shifting, we obtain

$$S_n = \sum_{0\le k\le n}\sum_{0\le m\le k}\binom{n}{k}\binom{n}{m} + \sum_{0\le k\le n}\sum_{0\le m\le k-1}\binom{n}{k}\binom{n}{m}$$

Thus $S_n = 2T_n - U_n$, where

$$T_n = \sum_{0\le k\le n}\sum_{0\le m\le k}\binom{n}{k}\binom{n}{m}, \quad U_n = \sum_{0\le k\le n}\binom{n}{k}^2.$$

Now we examine the sum T_n and see that also

$$T_n = \sum_{0\le m\le n}\sum_{m\le k\le n}\binom{n}{k}\binom{n}{m}.$$

Renaming the variables, we obtain

$$T_n = \sum_{0\le k\le n}\sum_{k\le m\le n}\binom{n}{k}\binom{n}{m}.$$

Adding this to the original expression for T_n gives $2T_n = V_n + U_n$, where

$$V_n = \sum_{0\le k\le n}\sum_{0\le m\le n}\binom{n}{k}\binom{n}{m}.$$

But in V_n, the bounds of the two sums have become decoupled, so we can write

$$V_n = \left(\sum_{0\le k\le n}\binom{n}{k}\right)\left(\sum_{0\le m\le n}\binom{n}{m}\right) = 2^n.2^n = 2^{2n}.$$

In conclusion, $S_n = V_n = 2^{2n}$.